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# A Dark Energy Model in Kaluza-Klein Cosmology

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**Abstract** We study a dynamic  $\Lambda$  model with varying gravitational constant  $G$  under the Kaluza-Klein cosmology. Physical features and the limitations of the present model have been explored and discussed. Solutions are found mostly in accordance with the observed features of the accelerating universe. Interestingly, signature flipping of the deceleration parameter is noticed and the present age of the Universe is also attainable under certain stringent conditions. We find that the time variation of gravitational constant is not permitted without vintage  $\Lambda$ .

**Keywords** Einstein's general relativity · Kaluza-Klein cosmology ·  $\Lambda$  dark energy

## 1 Introduction

The cosmological picture of an accelerating Universe which came into lime-light through seminal papers of Perlmutter et al. [1] and Riess et al. [2] is, at present, a well established scientific truth. The accelerating agent is termed as dark energy and the vintage cosmological term  $\Lambda$  is a favourite choice of

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the researchers working with Lambda-dark energy models as the dark energy representative. Although Einstein [3] introduced  $\Lambda$  as a constant in his field equations, due to Cosmological Problem and Coincidence Problem, now-a-days in most of the cases it is considered as a dynamical quantity [4]. On the other hand, the gravitational constant  $G$  is, in general, taken as a constant. However, sufficient amount of works related to variability of  $G$ , both at theoretical and experimental front, are also available in literature [5,6].

In the third decade of the previous century, Kaluza [7] and Klein [8] attempted to unify electro-magnetic force with gravitational force which resulted in the development of Kaluza-Klein (KK) theory. In KK approach, an extra dimension, viz. fifth dimension was introduced for coupling the two forces mentioned earlier. Chodos and Detweiler [9] have shown in their five dimensional model that the extra dimension contracts due to cosmic evolution. According to Guth [10] and Alvarez and Gavela [11], production of huge entropy due to the presence of an extra dimension can solve the flatness and horizon problems without invoking the idea of inflation. So, five dimensional model in the framework of KK theory has been successful in addressing some of the problematic issues of Big Bang cosmology and other realm of physics (for a critical review on the KK theory one can look at the new-born article by Wesson [12]).

A number of works are available in the literature where one or both of the cosmological constant  $\Lambda$  and the gravitational constant  $G$  are assumed to be variable. Pradhan et al. [13] and Ozel et al. [14] worked in the framework of KK cosmology with variable  $\Lambda$  and constant  $G$  while Mukhopadhyay et al. [6] performed an  $n$ -dimensional investigation with both  $\Lambda$  and  $G$  as variables. Sharif and Khanum [15] and more recently Oli [16] have worked with KK cosmological models by considering both  $\Lambda$  and  $G$  as variables. A special note to the work of Sharif and Khanum [15] is that they have investigated the effect of  $\Lambda = \epsilon H^2$  model (where  $\epsilon$  is a parameter) in KK cosmology with variable  $\Lambda$  and  $G$ . Ray et al. [5] showed the equivalence of  $\Lambda \sim H^2$ ,  $\Lambda \sim \ddot{a}/a$  and  $\Lambda \sim \rho$  models in the four dimensional FRW spacetime. So, the major motivations of the present work is to explore those features which were not touched upon in the work of Sharif and Khanum [15]. Although generalized energy conservation law for variable  $\Lambda$  and  $G$  models have been derived by Shapiro et al. [17] and Vereschagin et al. [18] in two different ways, Shapiro et al. [17] have mentioned in their work that without any loss of generality usual energy conservation law can be retained for variable  $\Lambda$  and variable  $G$  models. Beesham [19] has supported this idea of retention of usual energy conservation law by stating that the usual energy conservation law can be used within a simple framework of variable  $\Lambda$  and  $G$  [20,21,22,23,24].

The paper is organized as follows. Section 2 deals with field equations and their solutions; physical features explored in the work are presented in Section 3 while some discussions are done in Section 4.

## 2 Field equations and their solutions

The metric of KK cosmology is given by

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 + (1 - kr^2) d\psi^2 \right] \quad (1)$$

where  $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ .

Perfect fluid energy-momentum tensor takes the usual form as

$$T_{ij} = (p + \rho)u_\mu u_\nu - pg_{\mu\nu}, \quad (2)$$

where  $\mu, \nu = 0, 1, 2, 3, 4$  and  $u_\mu$  are five-velocity satisfying  $u_\mu u_\nu = 1$ ,  $\rho$  is the energy density and  $p$  is the pressure of the cosmic fluid.

The Einstein field equations are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = 8\pi GT_{ij}, \quad (3)$$

where the terms used have their usual meanings.

Using (1) - (3) we have the following two equations

$$8\pi G\rho + \Lambda = 6 \left( \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right), \quad (4)$$

$$8\pi Gp - \Lambda = -3 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right). \quad (5)$$

For a flat Universe,  $k = 0$  and hence Eqs. (4) and (5) reduce to

$$8\pi G\rho + \Lambda = 6 \frac{\dot{a}^2}{a^2} = 6H^2, \quad (6)$$

$$8\pi Gp - \Lambda = -3\dot{H} - 6H^2, \quad (7)$$

where  $H = \dot{a}/a$  is the Hubble parameter.

The continuity equation yields

$$\dot{\rho} + 4H(p + \rho) = 0, \quad (8)$$

while the equation of state is

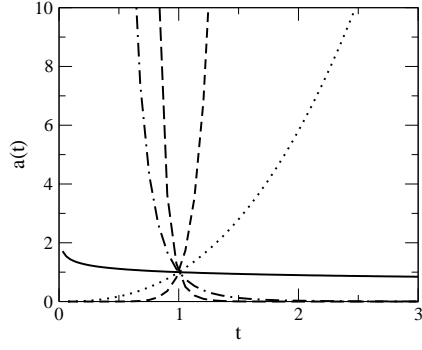
$$p = \omega\rho, \quad (9)$$

where  $\omega$  is the barotropic index.

Let us use the ansatz  $\Lambda = \beta(\ddot{a}/a) = \beta(\dot{H} + H^2)$ , where  $\beta$  is a parameter. This type of model was dealt by Overduin and Cooperstock [4] and Arbab [25, 26, 27] to account for the accelerating phase of the present Universe.

Then equation (6) becomes

$$\beta\dot{H} + (\beta - 6)H^2 = -8\pi G\rho. \quad (10)$$



**Fig. 1** Plot for variation of scale factor,  $a$ , with respect to age of the Universe,  $t$ , as given by equation (13). The solid, dotted, dashed, long-dashed and dot-dashed curves represent  $\omega = 0, -0.7, -0.9, -1.1$  and  $-1.3$ , respectively.

Also equation (7) reduces to

$$(\beta - 3)\dot{H} + (\beta - 6)H^2 = 8\pi Gp. \quad (11)$$

Dividing (11) by (10) and then simplifying we get

$$\frac{\dot{H}}{H^2} = -\frac{(1 + \omega)(\beta - 6)}{(\beta + \beta\omega - 3)}, \quad (12)$$

which on integration yields the solution set as

$$a(t) = Ct^{\frac{\beta + \beta\omega - 3}{(1 + \omega)(\beta - 6)}}, \quad (13)$$

$$H(t) = \frac{\beta + \beta\omega - 3}{(1 + \omega)(\beta - 6)t}, \quad (14)$$

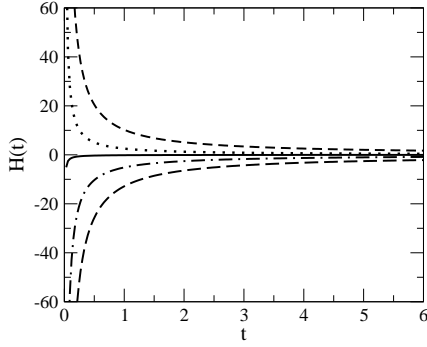
$$\Lambda(t) = \frac{3\beta(1 + 2\omega)(\beta + \beta\omega - 3)}{(1 + \omega)^2(\beta - 6)^2t^2}, \quad (15)$$

$$\rho(t) = C_1 t^{\frac{-4(\beta + \beta\omega - 3)}{(\beta - 6)}}, \quad (16)$$

$$G(t) = \frac{3}{8\pi C_1} \frac{\beta + \beta\omega - 3}{(1 + \omega)^2(\beta - 6)} t^{\frac{2\beta(1 + 2\omega)}{\beta - 6}}, \quad (17)$$

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -\frac{3 + 6\omega}{\beta + \beta\omega - 3}. \quad (18)$$

To obtain the present-day feature of the dust filled Universe one should put the constraint  $\omega = 0$  in the general form of Eqs. (13)-(18).



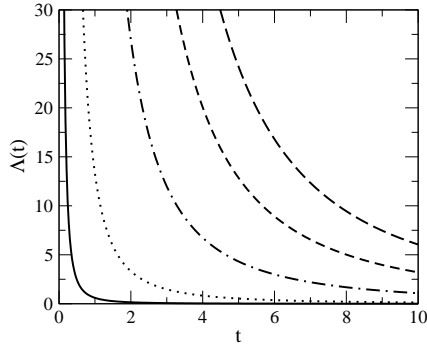
**Fig. 2** The description of various curves for Hubble parameter is the same as in Fig. 1.

### 3 Physical Features

The physical features of the present model in the framework of Kaluza-Klein theory are buried in its solution set through Eqs. (13) - (17). We choose here the numerical value of  $\beta$  from the range  $3.417 \leq \beta \leq 4.674$  in its lower limit i.e.  $\beta = 3.4$  [27, 28], which represents the observed accelerated expansion, and then plot these solutions for different values of  $\omega = 0.0, -0.7, -0.9, -1.1$  and  $-1.3$  as represented by solid, dotted, dashed, long-dashed and dot-dashed curves in Figs. 1-5. We take here, for simplicity,  $C = C_1 = 1$  in Eqs. (14), (16) and (13), where they appear as a multiplier or divider.

The obvious physical features are discussed below:

- (i) We find the condition,  $\beta \neq 6$ , for any physical validity of the solutions.
- (ii) For the values,  $\beta = 0$  or  $\omega = -0.5$  or  $\beta = 3/(1 + \omega)$ ,  $\Lambda$  becomes zero. Here, the last two conditions,  $\omega = -0.5$  and  $\beta = 3/(1 + \omega)$ , yield the unphysical situation,  $\beta = -6$ .
- (iii) Also, along with  $\Lambda$  the Hubble parameter reduces to zero with  $\beta = 3/(1 + \omega)$  while both the scale factor and the matter-energy density become constant. None of these features support the present cosmological scenario.
- (iv) With  $\beta = 0$ ,  $G$  becomes constant. Thus the present model does not permit time variation of  $G$  in the absence of the cosmological parameter  $\Lambda$ .
- (v) We also find  $t = 1/2H$  from Eq. (14) with  $\beta = \omega = 0$ . With the present value of the Hubble parameter as  $72 \text{ kms}^{-1} \text{ Mpc}^{-1}$ , the age of the Universe comes out to be 7 Gyr which is much below the accepted range of 13.2 to 14 Gyr (see Table 2 of the Ref. [28]). If we want to retain the non-zero values of  $\beta$  and  $\omega$  such that 3.4 and  $-0.36$  respectively then the above value of the age, viz. 7 Gyr can be obtained. However, to attain the modern-day accepted value  $\sim 14$  Gyr one has to put  $\omega = -0.5$ . Thus it is observed that with proper tuning of  $\beta$  one may recover the age of the Universe for any value of  $\omega$ . In this regard we would also like to make a general comment that all the above discussion justify the necessity of the inclusion of  $\Lambda$  in the field equations.



**Fig. 3** The description of various curves for Cosmological parameter is the same as in Fig. 1.

(vi) We also find inverse square law between time and density,  $\rho \sim t^{-2}$ , for  $\omega = 0$ . This relationship of  $\rho$  with cosmic time was obtained earlier by Ray et al. [28,29] in four dimensional model with  $\Lambda$ .

(vii) A negative value of the deceleration parameter,  $q$ , signifies the present accelerating phase. For the given value  $\beta = 3.4$ , the deceleration parameter  $q$  (Eq. 18) flips the sign and becomes positive if  $-0.118 > \omega > -0.5$ . For the dust filled case ( $\omega = 0$ ),  $q$  flips the sign if  $\beta < 3$ . A significant aspect of the evolution of the Universe is the flip over from a previously decelerating phase to the present accelerating one [30,31,32]. So, the deceleration parameter  $q$  must show a signature flipping to indicate that turnover and hence evolution of the Universe.

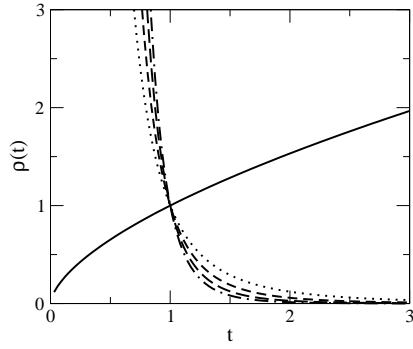
(viii) All the plots for  $a(t)$ ,  $H(t)$ ,  $\Lambda(t)$ ,  $\rho(t)$  and  $G(t)$  are found sensitive with the variation of  $\omega$  as it stands in the exponent of time and also due to the  $1 + \omega$  factor in the denominator.

There is a little variation for the scale factor,  $a(t)$ , at  $t = 1$  irrespective of the value of  $\omega$ . However, there is a large variation in the values of  $a(t)$  with respect to time. For the Hubble parameter, we find large or small value depending on  $\omega$ , which tend to zero with increasing time.

The  $\Lambda$  parameter is found to be always positive for the given  $\beta$  and  $\omega$  and thus acts as an dark energy agent of the accelerating scenario of the Universe.

Like  $a(t)$ , there is not much variation in the density,  $\rho$ , at  $t = 1$ . For  $\omega=0$ , density is found increasing with time. However, it shoots up at smaller time for non-zero values of  $\omega$ . This type of feature has been observed by Ray et al. [28,33] for which they made a comment that idea of inflation is inherent in the phenomenological model  $\Lambda = \beta(\ddot{a}/a) = \beta(\dot{H} + H^2)$ .

In the present investigation the gravitational parameter  $G$  varies with time and is also found sensitive with  $\omega$ .

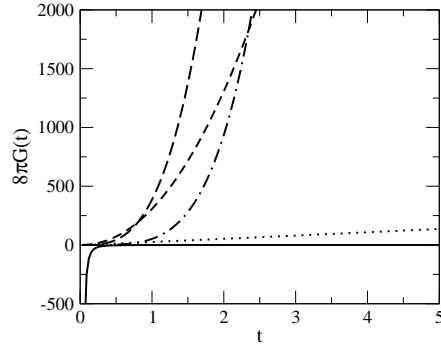


**Fig. 4** The description of various curves for the cosmic matter-energy density parameter is the same as in Fig. 1.

#### 4 Conclusions

In the present work we study cosmic evolution under the framework of Kluza-Klein cosmology. The emphasis has been on dynamic  $\Lambda$  model with varying gravitational constant. We present our solutions for the scale factor, Hubble parameter, gravitational constant and density with respect to time  $t$ . These are also plotted for various values of equation of state parameter,  $\omega$ . Interesting physical features of these solutions have been summarized and discussed. Solutions are mostly found in accordance of the observed features of the Universe. We find that the time variation of the gravitational constant is not permitted without inclusion of  $\Lambda$ . Signature flipping of the deceleration parameter is attainable. Besides, some previous results are also recovered in the present work. For instance, relationship of both the matter-energy density [28,29] and gravitational constant [5] can be recovered from Eqs. (16) and (17) respectively. The present work therefore can be regarded as a continuation of two previous works [5,6] related to variation of the gravitational constant  $G$  and also confirmation of some of the earlier results [28,29].

However, in the present phenomenological investigation of simple and elegant type some of the crucial issues have not been addressed, such as: *Firstly*, calculation of the evolution of linear perturbations in this model. This point is particularly important, as the expansion laws are easy to match with observations, and distinguishing between models requires information on the perturbations. *Secondly*, one can raise a simple question such that where is the fifth dimension now? Actually Eq. (1) suggests that the fifth dimension is large and is around us. Therefore, there must be a mechanism to make it small as modern physicists worry a lot about how to make the extra-dimensions compactified and one can again look at the review on KK theory by Wesson [12] to get a possible way out from the problem. All these aspects can, for the time being, be left as future efforts.



**Fig. 5** The description of various curves for gravitational parameter is the same as in Fig. 1.

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